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Dynamics and stability in retail competition

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Abstract

Retail competition today can be described by three main features: (i) oligopolistic competition, (ii) multi-store settings, and (iii) the presence of large economies of scale. In these markets, firms usually apply a centralized decisions making process in order to take full advantage of economies of scales, e.g. retail distribution centers. In this paper, we model and analyze the stability and chaos of retail competition considering all these issues. In particular, a dynamic multi-market Cournot–Nash equilibrium with global economies and diseconomies of scale model is developed. We confirm the non-intuitive hypothesis that retail multi-store competition is more unstable than traditional small business that cover the same demand. The main sources of stability are the scale parameter, the number of markets, and the number of firms.

Keywords: Multi-market oligopoly; Cournot–Nash competition; Economies of scale; Stability; Chaos

1. Introduction

In an oligopolistic setting under a Cournot scheme [25], the strategy of each economic player depends on its own quantity decision, and on its rival's reaction. Puu was one of the first to explicitly show the complex dynamics of the oligopolistic setting under simple assumptions (isoelastic demand function and constant marginal cost) for two and three players [42–44]. This kind of analysis has grown significantly during the last decade in both, the mathematics and complex systems literature, as well as in the economically-oriented journals.

Indeed, since the Puu's approach, several games has been developed for the study of the market stability, focusing on different demand or price function [9,39], number of players [5,46], behavioral assumptions (naive [24,37,19], versus adaptive [13,18], bounded rationality [3,39,52] or heterogeneous expectations [1,2,8,36]). In terms of the cost function definition, several developments has been proposed as well as non-linear cost function [52,28,39,36,18], capacity constraints | [48,47,18,34] and some spillover effects [15,14,17,16].

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Most works in this line of research have concentrated in single markets with linear production structures (i.e. assuming constant returns to scale). Nevertheless, oligopolistic competition today seems to present multi-market phenomena and, in some cases, they showcase important economies of scale, especially in the retail industry. Indeed, supermarket chains and retailers of food, gasoline, supplies and services all compete for market share through multi-store formats over geographically separated markets. This localized competition is presented in different levels: city, region, or country. In this context, companies segment their strategies, tailoring their selected outcome for different types of consumers and competitors, which vary by geographical location. On the other hand, on the supply side, multi-market retailers usually try to take full advantages of their size, in other words, their economies of scale. For instance, through the development of distributions centers that attend most of the stores in a specific territory. Thus, as the cost structure of multi-market retails depends on the total volume of the produced goods, the individual cost structure of each store is usually coupled with the whole business. It is important to point out that, this system of production, implies a centralized decisions making process, which becomes in practice an extremely difficult task. Summing up, retail competition today can be described by three main features: (i) oligopolistic competition, (ii) multi-store setting, and (iii) the presence of economies of scale.

Applications of the Cournot scheme into the multi-market problem have been proposed before by economists, for example, in the case of international trade. Some of these works modeled the presence of economies of scale, for the domestic and foreign markets, considering the size (quantity produced) and other properties of firms [20,21,33,26,32,11,35,38]. Thus, for instance in a work of Krugman [33], a multi-market Cournot model with economies of scale was used to explain the successful performance of Japan as an exporting country at the beginning of the 1980s.

In theoretical terms, the multi-market oligopoly framework was revisited and generalized in the seminal paper by Bulow, Geanakoplos and Klemperer [22]. One of their main remarks is that the presence of a multi-store firm in a market may affect the position of the firm in other markets due to the existence of demand and/or supply spillovers. In the same line, Bernheim and Whinston, [12], show that with scale economies, the multi-market contact may produce "spheres of influence" [29], that occurs when each of the multi-market competing firms may be more efficient in some subset of these markets and less efficient in others (symmetric advantage) or when one firm is more efficient in all markets (absolute advantage). Despite these multi-market analysis, this literature has focused mainly on the demand side of the problem, not the supply side. Specifically, they refer to multi-market contact, when demands curves recognize substitution and complementarity of different products.

In terms of the analysis of the dynamics of the multi-market Cournot problem, we found only a few papers [51,6,7,40], focusing on different products and scope.

In this context, this research deals with the analysis of stability and chaos of multi-market competition in the presence of economies and diseconomies of scale, extending in this way the analysis of the dynamics of the oligopolistic competition. Thus, we model the main characteristics of the retail competition today, analyzing the dynamics and stability of this particular business system, and we compare these results with the stability analysis of traditional small business that cover the same demand, the classic formulations of Theocharis, Fisher and Puu.

The main hypothesis of the paper is that non-linear cost structures in multi-market setting are important sources of instability in the game outcome. Particularly, we study the stability of a multi-market Cournot–Nash equilibrium with global economies of scale, that is, the scale level that is related to the total production of firms, in all markets, as opposed to local economies of scale presented at each store individually or linear production structures. In this setting, the internal organization of a firm may affect its performances over the markets and the global equilibrium [10]. For example, multi-market firms that buy their products in a centralized manner, storing them in a distribution center, to be redistributed afterwards to their retailers store in all markets usually operate this way to obtain economies of scale in the process of buying and distribution. In this paper, we assume this type of centralized structure where companies takes advantage of their size, under economies of scale, that allow them to decrease their cost structure [49].

This paper is organized as follows: in Section 2, classical models of the dynamics of the Cournot problem are described. In Section 3, a Multi-market-Cournot problem is presented, considering interrelated cost structures and economies of scale. In Section 4, the study of the stability of the system is addressed and generalized for different numbers of market, competitors and level of economies of scale. In Section 5, the complex dynamics of the multi-market retail model is analyzed using numerical simulations. The numerical results of the dynamics are compared with those of the single-market classical models, using path graphics and bifurcation diagrams. Finally, the main conclusions for this work are presented.

2. Baseline: single market oligopoly models

The well-known Theocharis paper and those following it [50,41,45,23] proposed a Cournot Oligopoly model with inverse lineal demand function and constant marginal cost, that is,

$$P = a - \sum q^i \tag{2.1}$$

$$C^i = c^i q^i \tag{2.2}$$

where $i = 1, \ldots, N$ identify the player, P is the price of the good, C is the total cost and q is the quantity.

The profit is obtained subtracting the revenues by total cost:

$$\pi^i = q^i P\left(q^1, \ldots, q^N\right) - c^i q^i.$$

The optimization problem (maximization of individual profit) gives the following n best response (or reaction) functions:

$$q^i = \frac{a - c^i}{2} - \sum_{k \neq i} \frac{q^k}{2}. \tag{2.3}$$

With solution (Cournot–Nash equilibrium):

$$\overset{*}{q^i} = \frac{a - c^i + N\left(\bar{c} - c^i\right)}{N + 1}$$

with $\bar{c} = (1/N) \sum_i c^i$.

In order to transform the static game (2.3) into a dynamic one, the Cournot or naive strategy is used (see Cournot [25] and Puu [42]).[1] Thus, the long run map as an iterative process is given by

$$q^i(t+1) = \frac{a - c^i}{2} - \sum_{k \neq i} \frac{q^k(t)}{2}. \tag{2.4}$$

The dynamical system of N reaction functions defined by (2.4) has a stable equilibrium (fix point) for $n \leq 2$. For $n = 3$ the equilibrium is neutrally stable (stationary oscillations) and for $n \geq 4$ the equilibrium becomes unstable.

A generalization of the Cournot–Theocharis problem was developed by Fisher [31]. This research allows to work with increasing or decreasing returns to scale (i.e., economies or diseconomies of scale). In order to get this result, we need to add a non-linear term to the traditional linear cost function:

$$P = a - \sum_i q^i$$

$$C^i = c^i q^i + d\left(q^i\right)^2.$$

Thus, the profit takes the following form:

$$\pi^i = q^i P\left(q^1, \ldots, q^N\right) - c^i q^i - d\left(q^i\right)^2. \tag{2.5}$$

Some restrictions for avoiding non-negativity of outputs, price, profits and marginal cost are considered: $c > 0$, $a \geq c$ and $d > -1/2$.

Thus, the maximization of the profit (Eq. (2.5)) leads to the best response of the ith-firm:

$$q^i = \frac{a - \sum\limits_{k \neq i} q^k - c^i}{2(1 + d)}. \tag{2.6}$$

[1] In this strategy, it is assumed that a player at time t decides the next period production on the basis of the current rivals' output.

Then, the Cournot–Nash equilibrium is given by

$$\overset{*}{q^i} = \frac{\left(a - c^i\right)(1 + 2d) + N\left(\bar{c} - c^i\right)}{4d^2 + 2(N+2)d + N + 1}.$$

Hence, the naive dynamics takes the form

$$q^i(t+1) = \frac{a - \sum\limits_{k \neq i} q^k(t) - c^i}{2(1+d)}. \tag{2.7}$$

Finally, this dynamical system becomes stable when $(N-3)/2 < d$. Thus, if there are two players the game has a stable equilibrium if $d > -1/2$.

The approaches revised above (Theocharis and Fisher) were designed to model a single-market oligopoly problem (for example the rivalry between "mom-and-pop" stores). In this case, both the prices and the costs do not depend upon the behavior of the players in other markets, because they did not consider the case of large firms, with multiple operations in various locations.

In the next section, and taking as baseline the models presented above, we will develop a multi-market Cournot model with economies of scale that will allow us to describe the modern retail competition.

3. The model

Let us consider a multi-market oligopoly where N single-product firms compete in M markets. All markets are very far away, so there is no existence of arbitrage possibilities. Each market has its own price (from a linear demand function). Then, if q_j^i is the quantity of the ith company at the market j and Q_j the market supply, the selling price at the jth market is given by

$$P_j = a_j - \sum_i q_j^i = a_j - Q_j. \tag{3.1}$$

We assume a centralized managerial structure, where the production cost depends on the total outputs of each firm. Besides, owing to their size and specialization, the company can have economies of scale. So, we have

$$C^i = c^i \sum_j q_j^i + d\left(\sum_j q_j^i\right)^2 = c^i Q^i + d\left(Q^i\right)^2 \tag{3.2}$$

with c^i greater than zero. Depending on the value of d, the companies operate under economies of scale ($d < 0$) or diseconomies of scale ($d > 0$). The allowable range for the parameter d will be analyzed later.

The square form of the production cost was used previously for other scholars in a single-market context [31,27,39,53]. However, for this multi-market scheme, we use the non-linearity in order to couple the costs and to enable the existence of global economies (diseconomies) of scale. This is a more realistic approach for modern retail firms, because they usually produce and/or buy in large scale.

Under this cost structure, $c^i > 0$ and $dQ^i_{\max} > -c^i/2$, where $Q^i_{\max}$ is the maximum level of production of firm i. The theoretical maximum production (extreme case) is achieved when a company becomes a monopoly in all the markets (i.e. $Q_j = q_j^i,\ \forall j$). As $P_j \geq 0$, we have $\sum_j a_j > Q^i_{\max}$, and then we have $2d > c^i / \sum_j a_j$.

Thus, the profit function of each firm depends upon each market price, the quantity sold by the firm in that particular market, and the total cost of the firm, that considers the sum of the global cost and the local cost:

$$\pi^i = \sum_j q_j^i P_j - C^i \tag{3.3}$$

$$= \sum_j q_j^i \left(a_j - b_j \sum_i q_j^i\right) - c^i Q^i - d\left(Q^i\right)^2. \tag{3.4}$$

The managerial decision of each firm is to choose the quantity q_j^i that maximizes its global profits. In other words, the ith company which produces a total output of Q^i divided over the M different markets according to the output vector $\underline{Q}^i = \{q_1^i, q_2^i, \dots, q_M^i,\}$ needs to fix the optimal allocation of production given by $\overset{*}{\underline{Q}^i} = \left\{\overset{*}{q_1^i}, \overset{*}{q_2^i}, \dots, \overset{*}{q_M^i}\right\}$, where each one of the $\overset{*}{q_j^i}$ is a solution of the $M \times N$ simultaneous equations which comprising the first order conditions of this game given by

$$\frac{\partial \pi^i}{\partial q_j^i} = a_j - Q_j - q_j^i - c^i - 2dQ^i = 0. \tag{3.5}$$

Defining the residual market supply for i as $\tilde{Q}_j^i = Q_j - q_j^i$, and the participation of the firm i in other markets different to j as $\hat{Q}_j^i = Q^i - q_j^i$, the reaction function takes the form:

$$q_j^i = \frac{a_j - \tilde{Q}_j^i - c^i - 2d\hat{Q}_j^i}{2(1+d)}. \tag{3.6}$$

Clearly, the allocation decision depends on the decision of the other players on this market, and also depends on the participation of the firm in other markets (or the total firm supply). This result is consistent with the Cournot intuition and consistent to previous results for the single-market problem[2] [31,30,28].

In order to keep the game on rails, we have that $\overset{*}{q_j^i}$ is a maximum if and only if $d > -1$. Also the marginal profit for zero output must be non-negative [31], so $a_j \geq c^i, \forall i$. This condition joined with the previous restrictions gives

$$d > -\frac{1}{2M}. \tag{3.7}$$

Thus, the profit of each firm at the equilibrium point is given by

$$\pi^i = \sum_j \left(\overset{*}{q_j^i}\right)^2 + d\left(\sum_j \overset{*}{q_j^i}\right)^2. \tag{3.8}$$

4. Dynamic analysis of the equilibrium

For the dynamic modeling, we use naive expectations. Thus, the game is developed on discrete time as follows[3]:

$$q_j^i(t+1) = \frac{a_j - \tilde{Q}_j^i(t) - c^i - 2d\hat{Q}^i(t)}{2(1+d)}. \tag{4.1}$$

[2] When $\tilde{Q}^i = 0$.

[3] The dynamic proposed in (4.1) can lead to negative outputs without economic sense. Following [4], we will differentiate between two types of trajectories: admissible and feasible. Calling T the map defined by the NM equations of the game, the set of admissible (S) and feasible (F) points are defined, respectively, by

$$S = \left\{\left(q_1^1, q_2^1, \dots, q_M^1, q_1^2, q_2^2, \dots, q_2^M\right) \in R_+^{NM} : T^n\left(q_1^1, q_2^1, \dots, q_M^1, q_1^2, q_2^2, \dots, q_2^M\right) \in R^{NM}\ \forall n > 0\right\}$$
$$F = \left\{\left(q_1^1, q_2^1, \dots, q_M^1, q_1^2, q_2^2, \dots, q_2^M\right) \in R_+^{NM} : T^n\left(q_1^1, q_2^1, \dots, q_M^1, q_1^2, q_2^2, \dots, q_2^M\right) \in R_+^{NM}\ \forall n > 0\right\}.$$

Hence, the mathematical results are based on the admissible trajectories. However, in the economic context, only the feasible points will be considered.

The Jacobian matrix of the system is given by

$$J=\begin{bmatrix} \frac{\partial \underline{Q}^1(t+1)}{\partial \underline{Q}^1(t)} & \frac{\partial \underline{Q}^1(t+1)}{\partial \underline{Q}^2(t)} & \cdots & \frac{\partial \underline{Q}^1(t+1)}{\partial \underline{Q}^N(t)} \\ \frac{\partial \underline{Q}^2(t+1)}{\partial \underline{Q}^1(t)} & \frac{\partial \underline{Q}^2(t+1)}{\partial \underline{Q}^2(t)} & \cdots & \frac{\partial \underline{Q}^2(t+1)}{\partial \underline{Q}^N(t)} \\ \vdots & \ddots & & \vdots \\ \frac{\partial \underline{Q}^N(t+1)}{\partial \underline{Q}^1(t)} & \frac{\partial \underline{Q}^N(t+1)}{\partial \underline{Q}^2(t)} & \cdots & \frac{\partial \underline{Q}^N(t+1)}{\partial \underline{Q}^N(t)} \end{bmatrix} \tag{4.2}$$

where each one of the $N \times N$ entries of J is a $M \times M$ block matrix, that represents the change of the i-firm's reaction functions with respect to the previous outputs of k, with

$$\frac{\partial \underline{Q}^i(t+1)}{\partial \underline{Q}^k(t)}=\begin{bmatrix} \frac{\partial q_1^i(t+1)}{\partial q_1^k(t)} & \frac{\partial q_1^i(t+1)}{\partial q_2^k(t)} & \cdots & \frac{\partial q_1^i(t+1)}{\partial q_M^k(t)} \\ \frac{\partial q_2^i(t+1)}{\partial q_1^k(t)} & \frac{\partial q_2^i(t+1)}{\partial q_2^k(t)} & \cdots & \frac{\partial q_2^i(t+1)}{\partial q_M^k(t)} \\ \vdots & \ddots & & \vdots \\ \frac{\partial q_M^i(t+1)}{\partial q_1^k(t)} & \frac{\partial q_M^i(t+1)}{\partial q_2^k(t)} & \cdots & \frac{\partial q_M^i(t+1)}{\partial q_M^k(t)} \end{bmatrix}. \tag{4.3}$$

According to our optimal outputs (Eq. (4.1)), we have

$$\frac{\partial \underline{Q}^i(t+1)}{\partial \underline{Q}^k(t)}=\begin{cases} -\frac{1}{2(1+d)} I_{M\times M}, & \text{for } i \neq j \\ H, & \text{for } i=j \end{cases} \tag{4.4}$$

where I is the identity matrix and H is a zero-diagonal matrix with all the off-diagonal entries equal to $-d/(1+d)$.

Thus, when there are two players competing over M different markets, the long-run map proposed in (4.1) is defined for the following $2M$ equations:

$$\begin{aligned} q_1^1(t+1) &= \frac{a_1-q_1^2(t)-c^1-2d\left(q_2^1(t)+q_3^1(t)+\cdots+q_M^1(t)\right)}{2(1+d)} \\ &\vdots \\ q_j^1(t+1) &= \frac{a_j-q_j^2(t)-c^1-2d\left(q_1^1(t)+\cdots+q_{j-1}^1(t)+q_{j+1}^1(t)\ldots+q_M^1(t)\right)}{2(1+d)} \\ &\vdots \\ q_M^1(t+1) &= \frac{a_M-q_M^2(t)-c^1-2d\left(q_1^1(t)+q_3^1(t)+\cdots+q_{M-1}^1(t)\right)}{2(1+d)} \\ q_1^2(t+1) &= \frac{a_1-q_1^1(t)-c^2-2d\left(q_2^2(t)+q_3^2(t)+\cdots+q_M^2(t)\right)}{2(1+d)} \\ &\vdots \\ q_j^2(t+1) &= \frac{a_j-q_j^1(t)-c^2-2d\left(q_1^2(t)+\cdots+q_{j-1}^2(t)+q_{j+1}^2(t)\ldots+q_M^2(t)\right)}{2(1+d)} \\ &\vdots \\ q_M^2(t+1) &= \frac{a_M-q_M^1(t)-c^2-2d\left(q_1^2(t)+q_3^2(t)+\cdots+q_{M-1}^2(t)\right)}{2(1+d)}. \end{aligned} \tag{4.5}$$

The nontrivial Cournot–Nash equilibrium point for the previous set of equations (static solution) is given by

$$\overset{*}{q}{}^{i}_{j} = \frac{\left(a_j - c^i\right)(1 + 2Md) + c^k - c^i}{3 + (2M)^2 d^2 + 8Md} + \frac{2}{3} \frac{M\left(a_j - \bar{a}\right)\left(2Md^2 + d\right)}{3 + (2M)^2 d^2 + 8Md}, \quad d \neq -\frac{3}{2M}, -\frac{1}{2M}. \tag{4.6}$$

Using Eqs. (4.2)–(4.4), the Jacobian matrix for the dynamical system proposed in (4.5) is defined by

$$J_{2xM} = \left[\begin{array}{cc} M\text{times}\left\{\underbrace{\begin{matrix} 0 & -\frac{d}{1+d} & \cdots & -\frac{d}{1+d} \\ -\frac{d}{1+d} & 0 & \cdots & -\frac{d}{1+d} \\ \vdots & \ddots & & \vdots \\ -\frac{d}{1+d} & -\frac{d}{1+d} & \cdots & 0 \end{matrix}}_{M\text{times}}\right. & \begin{matrix} -\frac{1}{2(1+d)} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2(1+d)} & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2(1+d)} \end{matrix} \\ \begin{matrix} -\frac{1}{2(1+d)} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2(1+d)} & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -\frac{1}{2(1+d)} \end{matrix} & \begin{matrix} 0 & -\frac{d}{1+d} & \cdots & -\frac{d}{1+d} \\ -\frac{d}{1+d} & 0 & \cdots & -\frac{d}{1+d} \\ \vdots & \ddots & & \vdots \\ -\frac{d}{1+d} & -\frac{d}{1+d} & & 0 \end{matrix} \end{array}\right]. \tag{4.7}$$

The characteristic equation of (4.7) takes the form:

$$\left(\lambda - \frac{1}{2}\frac{2d+1}{(1+d)}\right)^{M-1}\left(\lambda - \frac{1}{2}\frac{2d-1}{(1+d)}\right)^{M-1}\left(\lambda + \frac{1}{2}\frac{2(M-1)d+1}{(1+d)}\right)\left(\lambda + \frac{1}{2}\frac{2(M-1)d-1}{(1+d)}\right) = 0. \tag{4.8}$$

Thus, we have four[4] different eigenvalues:

$$\begin{aligned} \lambda_1 &= -\frac{2(M-1)d+1}{2(1+d)} \\ \lambda_2 &= -\frac{2(M-1)d-1}{2(1+d)} \\ \lambda_{3,\dots,M+1} &= \frac{2d+1}{2(1+d)} \\ \lambda_{M+2,\dots,2M} &= \frac{2d-1}{2(1+d)}. \end{aligned} \tag{4.9}$$

The local stability of equilibrium is achieved only if each eigenvalue is within the unit circle. According to (4.9), this is fulfilled when

$$\begin{cases} -\frac{1}{2M} < d < \frac{1}{2(M-2)}, & M > 2 \\ d > -\frac{1}{2M}, & M = 1, 2 \end{cases}. \tag{4.10}$$

By (4.10), it is clear that the stability of the duopoly game depends on both the scale parameter (d)and the number of markets (M). In Fig. 4.1, the relationship between M and d, in order to arrive to the stability of the game is shown. When $M = 1, 2$ the equilibrium becomes unstable only if $d > -1/(2M)$. However, if $M \geq 3$, we need to consider an upper bound for the stability condition. We see that when the number of markets increases, the stability is achieved only if d tends to zero.

[4] If we fix $M = 1$, we have only the two first eigenvalues.

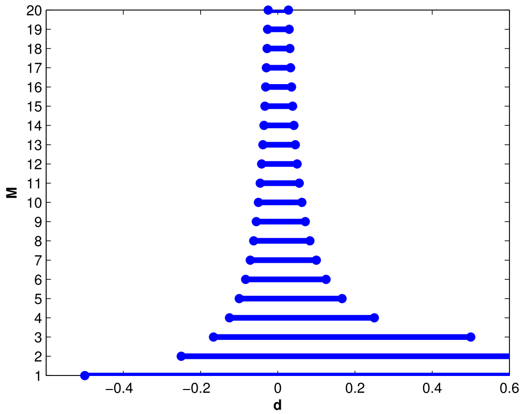


Fig. 4.1. Stability zone (lines) for a duopoly depending of the number of markets and the values for d.

Table 1
Stability conditions for the proposed model of retail competition and the baseline single market approaches.

	Cournot–Theocharis model	Fisher model	Proposed model of retail competition	
$N = 2$	Stable	Stable for $d > -0.5$	$M = 1$	Stable for $d > -0.5$
			$M = 2$	Stable for $d > -1/4$
			$M \geq 3$	Stable for $-1/2M < d < 1/2\,(M-2)$
$N = 3$	Neutrally stable	Stable for $d > 0$	$M = 1$	Stable for $d > 0$ Neutrally stable for $d = 0$
		Neutrally stable for $d = 0$	$M = 2$	Neutrally stable for $d \geq 0$
			$M \geq 3$	Neutrally stable for $d = 0$ Unstable for $d \neq 0$
$N \geq 4$	Unstable	Stable for $d > \frac{N-3}{2}$	$M = 1$	Stable for $d > (N-3)/2$ Neutrally stable for $d = (N-3)/2$
		Neutrally stable for $d = \frac{N-3}{2}$	$M \geq 2$	Unstable

Analogously to the duopoly case, if we consider a triopoly competition over M markets, the eigenvalues for this dynamical system are given by

$$\begin{aligned}\lambda_1 &= -\frac{(M-1)\,d+1}{(1+d)}\\ \lambda_{2,3} &= -\frac{2\,(M-1)\,d-1}{2\,(1+d)}\\ \lambda_{4,\dots,M+2} &= \frac{2d+1}{2\,(1+d)}\\ \lambda_{M+3,\dots,3M} &= \frac{d-1}{1+d}.\end{aligned} \tag{4.11}$$

These eigenvalues lead to a stable game for $M = 1$ and $d > 0$ (Fisher approach). When $d = 0$ and $M = 1$, the game is reduced to the Cournot–Theocharis case (neutrally stable). We arrive to perpetual oscillations for $M = 2$, for $d \geq 0$ and complex dynamics for $d < 0$. For $d \neq 0$ and $M \geq 3$ the system leads to a chaotic behavior.

When the number of players is four or more, we maintain the Fisher condition for stability in the case of $M = 1$, ($d > 0.5$). Finally, for $M > 1$ the system is unstable, for any value of d.

In Table 1, we summarize our results, comparing them with the results of the both baseline approaches: the Cournot–Theocharis model and the Fisher model. We see that in the Cournot–Theocharis model, the only parameter that affects the stability condition is the number of players (N), and the path of the equilibrium quantity is chaotic when the number of competitors is greater than three. Fisher introduces the scale parameter d, that allows stability while $(N-3)/2 < d$. But when we add stores or markets, as our model of retail competition suggests, the stability zone is reduced (to a less or equal magnitude that the results of Fisher), depending on the values of N, d and M. In fact, if $d \neq 0$, when we have more than 3 players and more than one market, the response of any player follows a complex dynamics. If $M = 1$, we recover the Fisher Model. On the other hand, if $d = 0$ we deal with an uncoupled multimarket model, and our retail competition model is reduced to the Cournot–Theocharis case.

5. Numerical simulations and discussion

Using numerical simulations, we can see how the complex dynamics of the equilibrium of the multi-market retail model depends on the number of players, the scale parameter (d) and the number of markets (M).

First, we use a scheme of duopolist who compete on three markets (2×3 game). The results of the retail model (blue) will be compared with the base model (red) proposed by Fisher (see Section 2) under identical parameter values. Figs. 5.1 and 5.2 show the dynamic of the quantity allocated by player 1 in market 1 according to several values of d.

For $d = -0.1$ and $d = 0.2$ (Figs. 5.1(b), 5.2(a), respectively), we have that the dynamics of the optimal values under naive expectations reaches the equilibrium in both models. When $d = 0$ (Fig. 5.1(c)), the models no longer have advantages/disadvantages of scale production. Hence, both models have the same schemes reflected in the Cournot–Theocharis approach in the $d = 0$ case (see Section 2). As seen in Fig. 5.2(b), for the critical point $d = 1/2$, the dynamic goes to stationary oscillations (neutrally stable) for the multi-market retail model, while the single-market model arrives quickly to the equilibrium. For the cases shown in Figs. 5.1(a), 5.2(c), the dynamic of the Fisher model remains stable, nevertheless the multi-market retail model presents chaotic behavior.

Fig. 5.3 shows the complex dynamic of the quantity q_1^1 by means of bifurcation diagrams, using values of d from the interval $[-0.17; 0.52]$. The decentralized model shows stability of the equilibrium for all the values examined. By contrast, for the multi-market retail model, chaos is found outside the critical points ($d < -1/6$ and $d > 1/2$), as it was predicted by the analytical results of the duopoly case, see Eq. (4.10). In the stability zone (Fig. 5.3(b)), we can observe how the centralized managerial decision takes advantage in terms of production when the company operates under economies of scale in relation to the disaggregated model; while at diseconomies of scale, the production performance of the local model are less affected than the multi-market scheme.

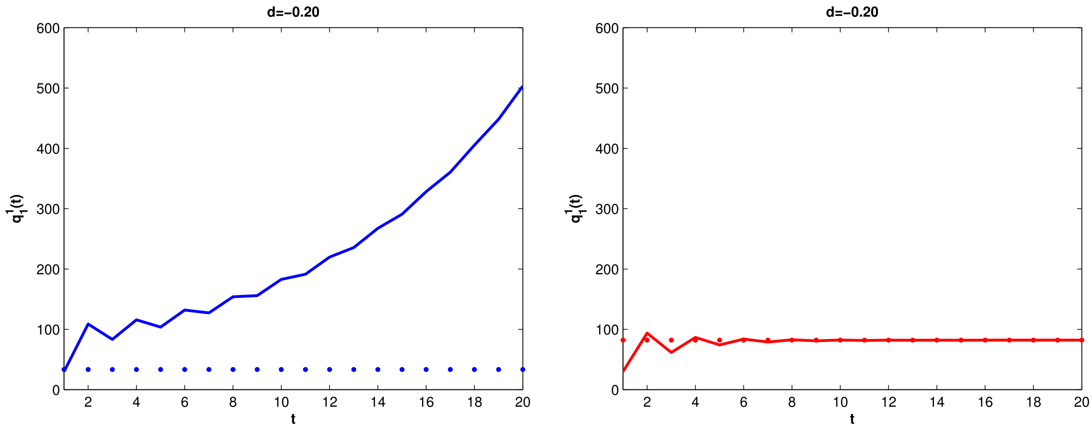
On the other hand, Figs. 5.4 and 5.5 show the results for three companies competing on one, two and three markets. The Fisher condition for stability sets is d must be greater than zero. However, when the number of markets is two, $M = 2$, the systems is neutrally stable for $d \geq 0$. For $M \geq 3$, the game becomes unstable for all the values of d. Finally, in Fig. 5.6, the dynamics of four players is shown, when $N = 4$. The system is stable for $d > 0.5$ for the Fisher condition. But when we add markets ($M \geq 2$), the system leads to complex dynamics.

These results show how the Nash equilibrium of the multi-store retail competition with global economies of scale presents a chaotic dynamics depending on the scale parameter, the number of markets, and the number of firms.

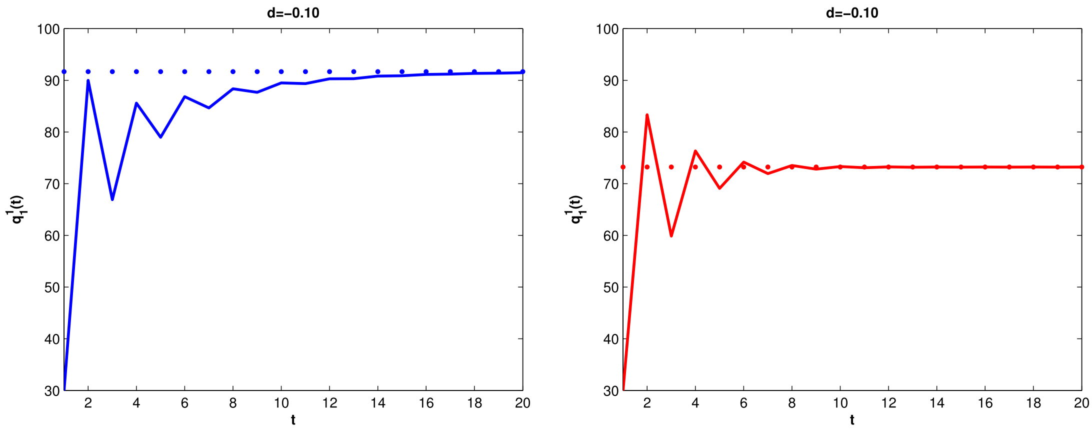
Indeed, firstly, the level of global economies of scale implies instability in the equilibrium for some ranges of the scale parameter. In other words, instability in the optimal level of production of each store is consistent with the findings of Fisher's classical work. Secondly, the number of stores introduces an additional source of complexity to the already chaotic dynamics of the traditional single-market, multi-player oligopoly, as the Cournot–Theocharis approach suggests. But what does this mean in terms of the managerial decision making process of retail?

In the multi-store retail competition problem, the central management has to decide how much to buy for each store, and hence how much to buy in total, since the buying power of the firm must be exercised in a centralized fashion with the providers. Given the fact that our model shows chaotic results for the optimal centralized decision making model, considering some levels of economies of scale and the level of local competition (store), one way to control chaos is to restrict the use of the model to the range of the scale parameter, when the system is stable. If this is not the case, the only way to control chaos is to point to less complex, sub-optimal, solutions.

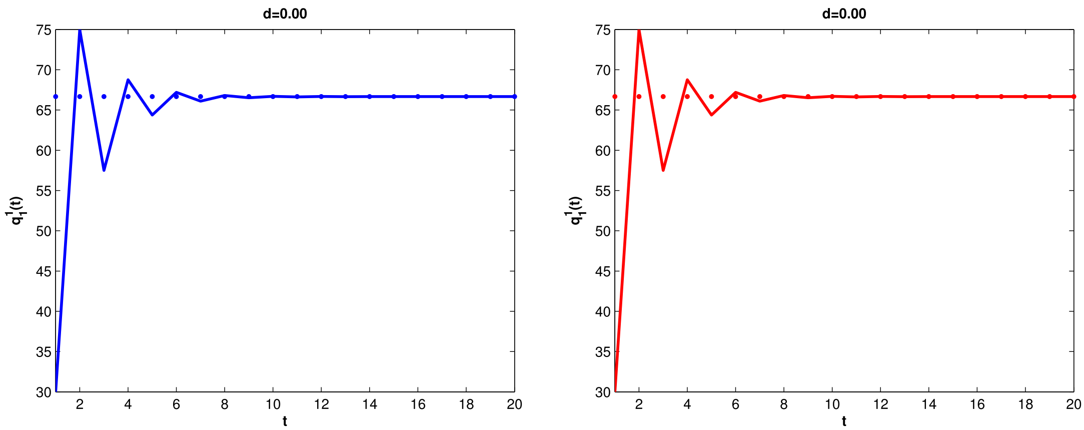
In the latter case, the central management faces two alternatives: (i) to model the problem in an aggregated way, considering the national oligopoly, with the national players, and the total amount of product, and taking into account the global economies of scale. With these results, the central management can later on to distribute the amount among stores proportionally to their sales, or (ii) to implement a decentralized decision making, leaving to the local management, the store manager, to find the optimal amount of production for its own store, considering the store's profit maximization problem, and the local level of competition. It is important to remark that in the latter alternative,



(a) $d = -0.2$.



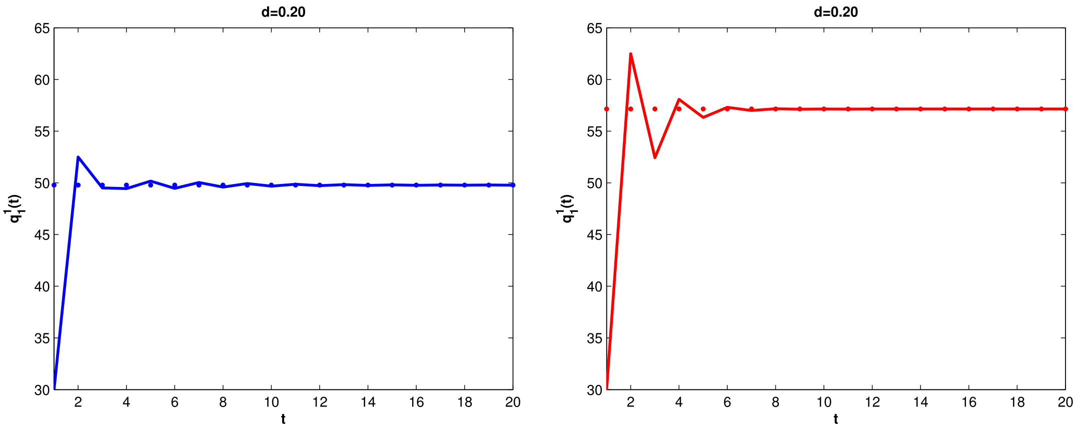
(b) $d = -0.1$.



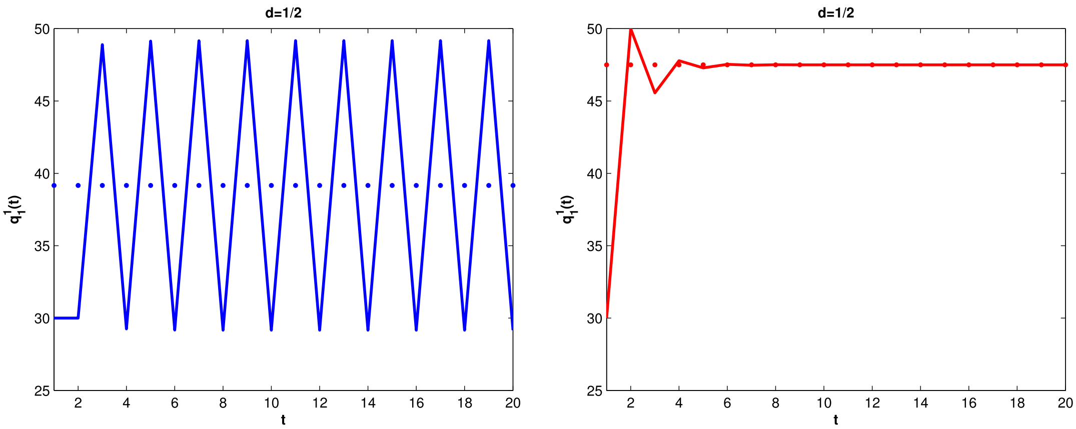
(c) $d = 0$.

Fig. 5.1. Dynamic of q_1^1 (solid line) and Cournot equilibrium (dots), for different values of d, under the proposed retail competition model (blue) and the Fisher approach (independent sellers) (red); with $a_1 = 200$, $a_2 = 150$, $a_3 = 100$, $c_1 = 20$, $c_2 = 40$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

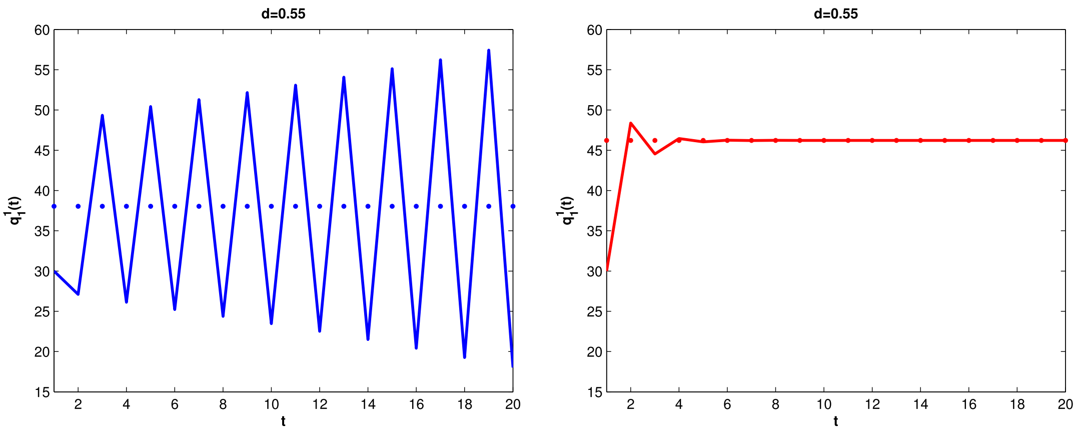
in practical terms some level of global economies of scale could also be achieved, when the local decisions are aggregated, but they are not considered explicitly in the optimization problem, so they are always sub-optimal.



(a) $d = 0.2$.



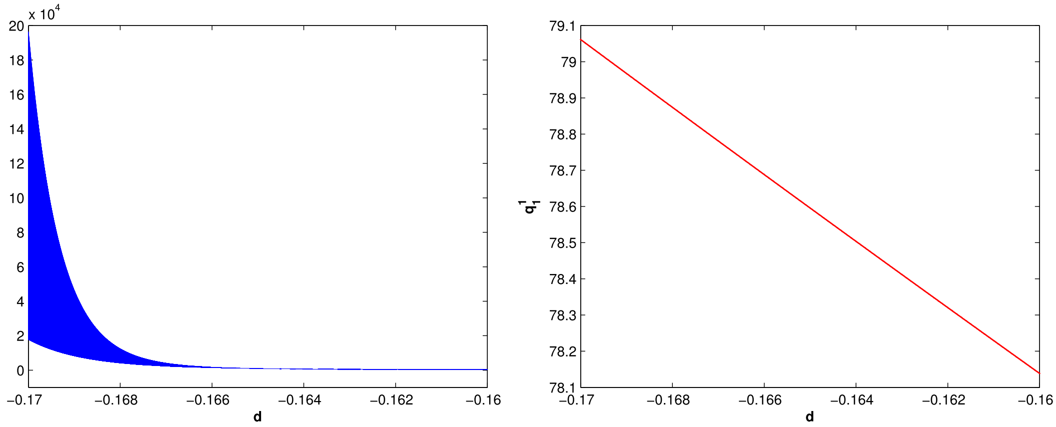
(b) $d = 0.5$.



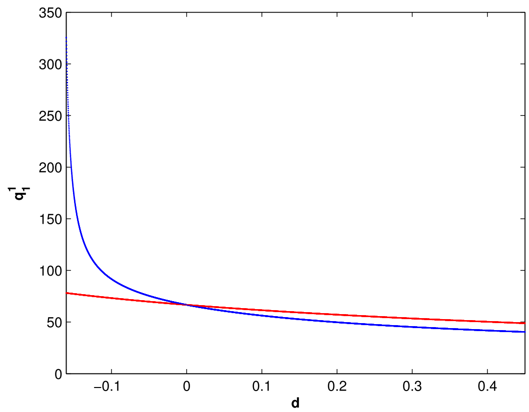
(c) $d = 0.55$.

Fig. 5.2. Dynamic of q_1^1 (solid line) and Cournot equilibrium (dots), for different values of d, under the proposed retail competition model (blue) and the Fisher approach (red); with $a_1 = 200$, $a_2 = 150$, $a_3 = 100$, $c_1 = 20$, $c_2 = 40$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

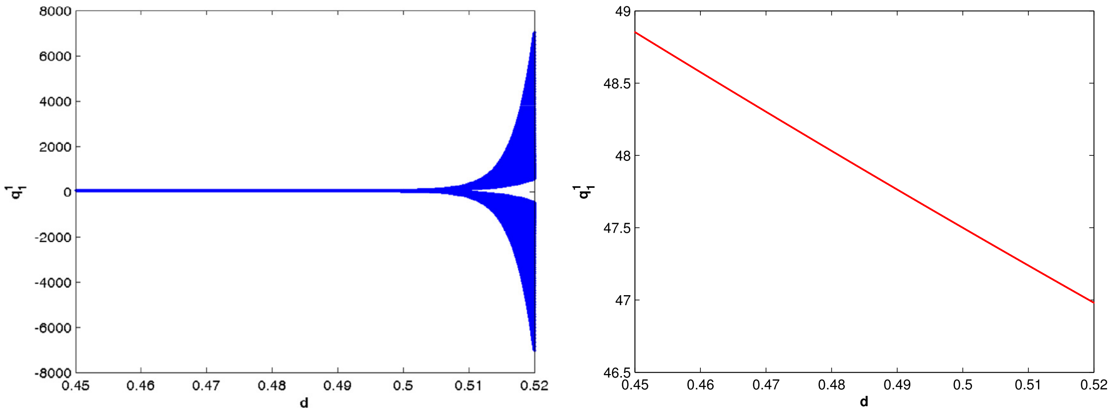
As we already know, (i) represents the Fisher model, while (ii) represents the Cournot–Theocharis approach. Which of these alternatives is better will depend upon the particular characteristics and parameters of the problem. In this



(a) $d \in [-0.17; -0.16]$.



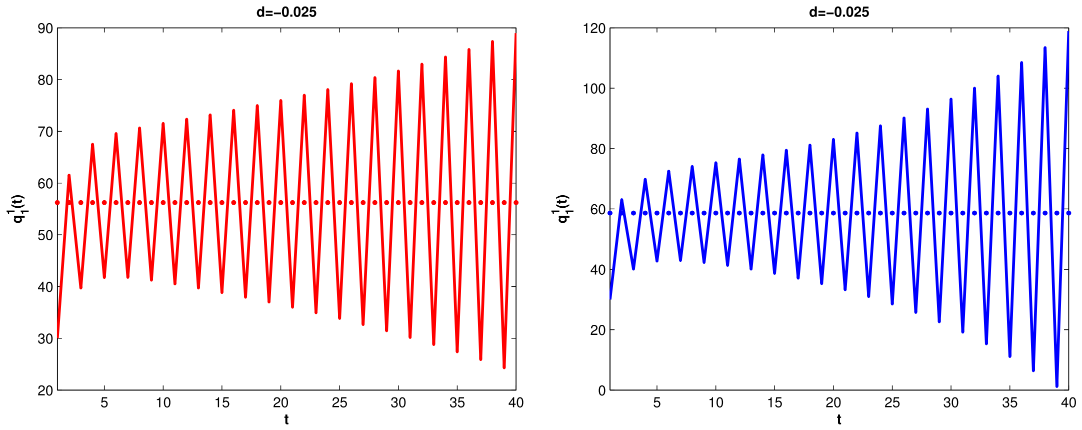
(b) $d \in [-0.16; 0.45]$.



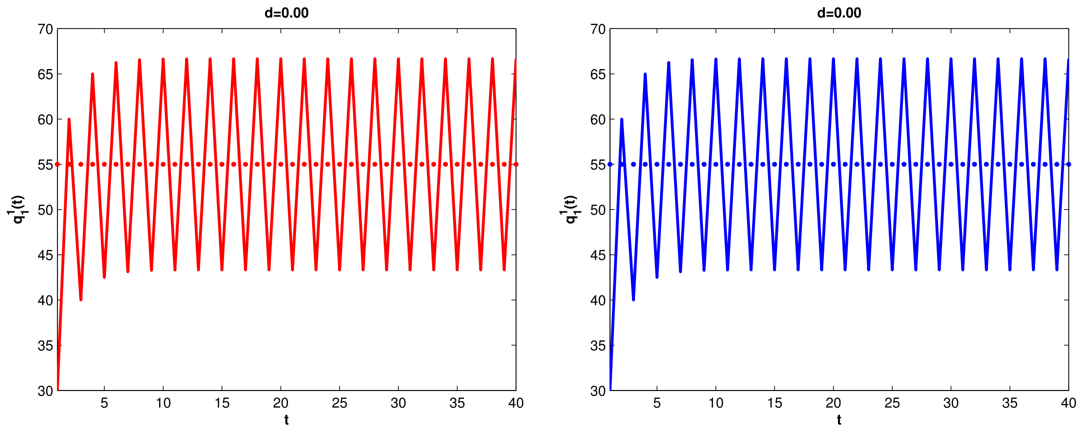
(c) $d \in [-0.45; 0.52]$.

Fig. 5.3. Bifurcation diagrams for the quantity of the player 1 over market 1 using the proposed retail competition model (blue) and the Fisher approach (red) with $a_1 = 200$, $a_2 = 150$, $a_3 = 100$, $c_1 = 20$ and $c_2 = 40$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

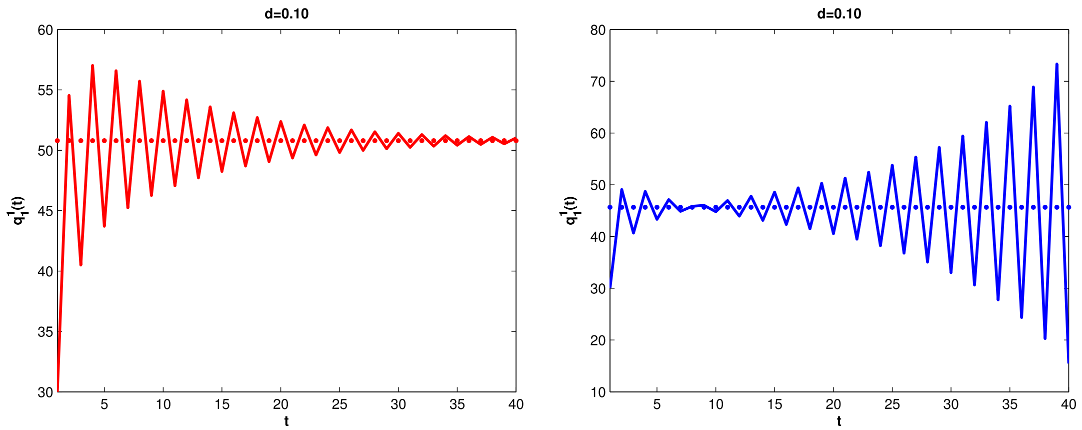
context, the modern big-box retailer, despite the advances of the management science and the computation power of computers, in some cases, must conform to a second-best world. Indeed, some of these second-best alternatives can be appreciated in the real world, for instance in the large multi-national retail franchising of clothes and food, where



(a) $d = 0.025$.



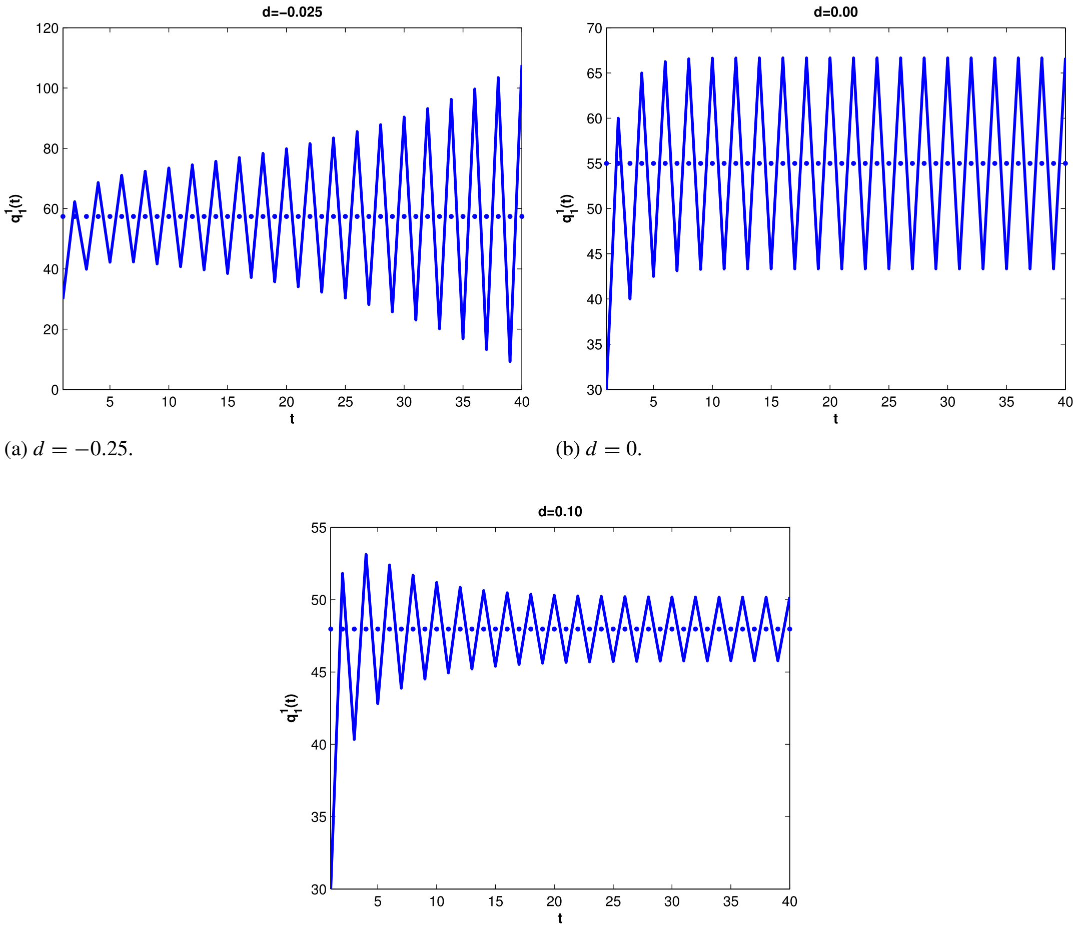
(b) $d = 0$.



(c) $d = 0.1$.

Fig. 5.4. Dynamic of q_1^1 (solid line) and Cournot equilibrium (dots), for 3 players, with different values of d, under the proposed retail competition model with 3 markets (blue) and the Fisher approach (independent sellers, red); with $a_1 = 200$, $a_2 = 150$ $c_1 = 20$, $c_2 = c_3 = 40$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

local managers make their own decisions of production, and the owner of the franchising take advantage of the buying and/or producer power, and then selling exclusively to every store manager.



(a) $d = -0.25$.

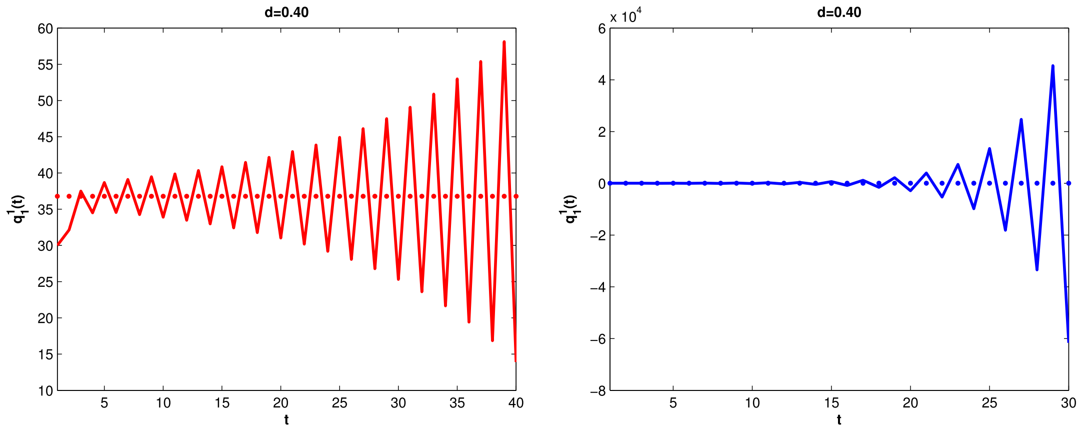
(b) $d = 0$.

(c) $d = 0.1$.

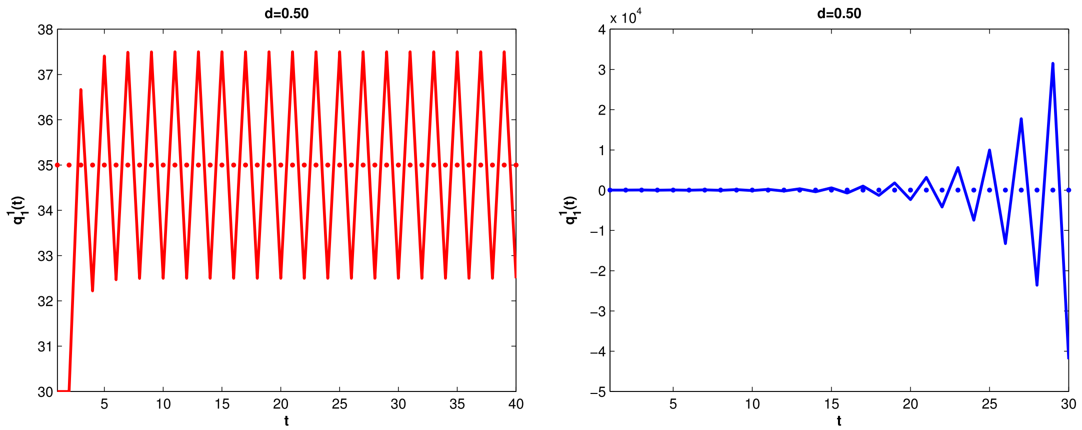
Fig. 5.5. Dynamic of q_1^1 (solid line) and Cournot equilibrium (dots), for 3 players and 2 markets, for different values of d, under the proposed retail competition model; with $a_1 = 200$, $a_2 = 150 c_1 = 20$, $c_2 = c_3 = 40$.

6. Conclusions

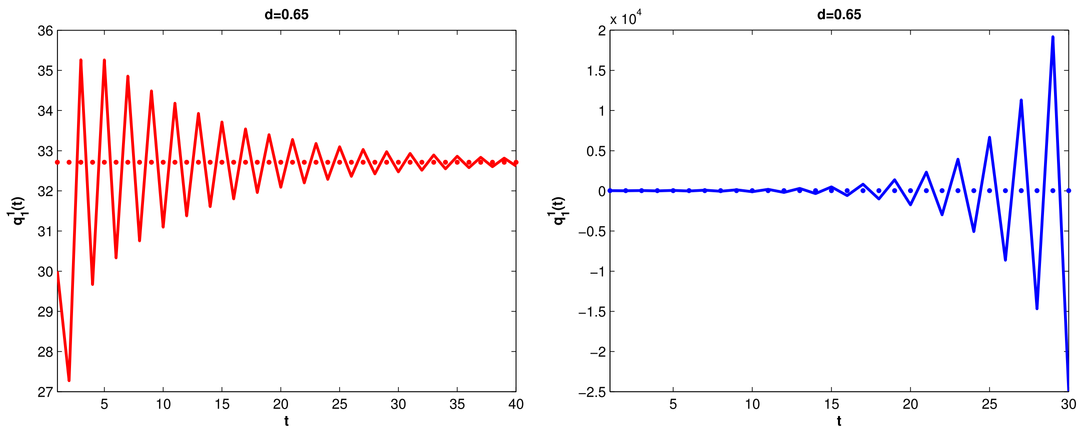
In this work, we analyze the stability of a multi-store retail competition model. Specifically, we model an oligopoly system with multi-market competition. Furthermore, the model considers the impact of the firms' economies or diseconomies of scale. In one extreme, the multi-store retail maintains a centralized decision making process, optimizing global economies of scale due to its global size. On the other hand, we have local oligopolistic competition, where the same demand is served by different firms that compete in only one market. Our model confirms the non-intuitive hypothesis that economies and diseconomies of scale make the Cournot equilibrium very unstable for certain values of the scale parameter of production. Additionally, the number of markets and competitors, as expected, tends to contribute to this instability. The implications of our results for the management of retail firms are that when chaos cannot be avoided in the decentralized multi-market retail model because of the scale range, managers should evaluate second-best models. Here, we have two options. Firstly, when decisions are taken in a centralized fashion, i.e. considering the totality of the production of all stores, managers should consider the Fisher approach, which incorporate economies and diseconomies of scale, in order to analyze the stability in the solution of the problem. On the other hand, when decision are taken in a decentralized way through for example local managers, the traditional Cournot–Theocharis approach should be applied to analyze the stability of this more simpler problem. It is important to remark that in the latter alternative, in practical terms, some level of global economies of scale could also be achieved when the local decisions are aggregated, but they are not considered explicitly in the optimization problem, so they are always sub-optimal. One very interesting further research is to expand the multi-market oligopolistic



(a) $d = 0.4$.



(b) $d = 0.5$.



(c) $d = 0.65$.

Fig. 5.6. Dynamic of q_1^1 (solid line) and Cournot equilibrium (dots), for 4 players, with different values of d, under the proposed retail competition model with 2 markets (blue) and the Fisher approach (independent sellers) (red); with $a_1 = 200$, $a_2 = 150$ $c_1 = 20$, $c_2 = c_3 = 40$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

model to a multi-product setting with economies of scope, and to compare these results in the context of Bertrand competition.

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